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Generalized Gravi—Electromagnetism

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Abstract A self consistant and manifestly covariant theory for the dynamics of four charges (masses) (namely electric, magnetic, gravitational, Heavisidian) has been developed in simple, compact and consistent manner. Starting with an invariant Lagrangian density and its quaternionic representation, we have obtained the consistent field equation for the dynamics of four charges. It has been shown that the present reformulation reproduces the dynamics of individual charges (masses) in the absence of other charge (masses) as well as the generalized theory of dyons (gravito-dyons) in the absence gravito-dyons (dyons).

Keywords Dyons · Gravito-dyons · Quaternion

1 Introduction

Magnetic monopoles [1, 2] were advocated to symmetrize Maxwell's equations in a manifest way that the mere existence of an isolated magnetic charge implies the quantization of electric charge. The fresh interests in this subject are enhanced with the idea of t' Hooft and Polyakov [3, 4] that the classical solutions having the properties of magnetic monopoles may be found in Yang-Mills gauge theories. The Dirac monopoles were elementary particles but the t' Hooft-Polyakov [3, 4] monopoles are complicated extended object having a definite mass and finite size inside of which massive fields play an important role in providing a smooth structure and outside it they vanish rapidly leaving the field configuration identical to abelian Dirac monopole. Julia and Zee [5] extended the theory of non-Abelian monopoles of t' Hooft-Polyakov [3, 4] to the theory of non Abelian dyons (particles carrying simultaneously electric and magnetic charges). The quantum mechanical excitation of fundamental

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monopoles include dyons which are automatically arisen from the semi-classical quantization of global charge rotation degree of freedom of monopoles. Despite of the potential importance of monopoles [1–4] and dyons [5], the formalism necessary to describe them has been clumsy and not manifestly covariant. So, a self consistent and manifestly covariant theory of generalized electromagnetic fields of dyons (particle carrying electric and magnetic charges) and those for generalized fields of gravitodyons, has been constructed [6–11] in terms of two four-potentials to avoid the use of controversial string variables. On the other hand,the quaternionic formulation [12–19] of electrodynamics has a long history [20–31], stretching back to Maxwell himself [20–22] who used real (Hamilton) quaternion in his original manuscript 'on the application of quaternion to electromagnetism' and in his celebrated book "Treatise on Electricity and Magnetism". Quaternion analysis has since been rediscovered at regular intervals and accordingly the Maxwell's Equations of electromagnetism were rewritten as one quaternion equations [32–46]. Negi and coworkers [47–50], have also studied the quaternionic formulation for generalized electromagnetic fields of dyons (particles carrying simultaneous existence of electric and magnetic charges) in unique, simpler and compact notations. Kravchenko and co-authors [51–54], discussed the Maxwell's equations in homogeneous media, chiral media and inhomogeneous media. Accordingly they have developed the quaternionic reformulation of the time-dependent Maxwell's equations along with the classical solution of a moving source, i.e. electron. In the series of papers Negi and coworkers [55–59] have derived the generalized Dirac-Maxwell (GDM) equations in presence of electric and magnetic sources in an isotropic (homogeneous) medium. They have also analyzed the other quantum equations of dyons in consistent and manifest covariant way [55]. This theory has been shown to remain invariant under the duality transformations in isotropic homogeneous medium. Quaternion analysis of time dependent Maxwell's equations has been developed by them [56] in presence of electric and magnetic charges and the solutions for the classical problem of moving charge (electric and magnetic) are obtained consistently. The time dependent generalized Dirac-Maxwell's (GDM) equations of dyons are also discussed [57] in chiral and inhomogeneous media and the solutions for the classical problem are obtained. The quaternion reformulation of generalized electromagnetic fields of dyons in chiral and inhomogeneous media has also been analyzed [58]. The monochromatic fields of generalized electromagnetic fields of dyons have also discussed [59] in slowly changing media in a consistent manner. The quaternion analysis has also been used [60] to combine the complex description of dyons and gravito-dyons and accordingly the unified quaternionic angular momentum for generalized fields of dyons and gravito-dyons along with their commutation relations has been analyzed in unique and consistent manner. In this paper, we have made an attempt to developed a self consistant and manifestly covariant theory for the dynamics of four charges (masses) (namely electric, magnetic, gravitational, Heavisidian) has been developed in simple, compact and consistent manner. Starting with an invariant Lagrangian density and its quaternionic representation, we have obtained the consistent field equation and equation of motion for the dynamics of four charges. It has been shown that the present reformulation reproduces the dynamics of individual charges (masses) in the absence of other charge (masses) as well as the generalized theory of dyons (gravito-dyons) in the absence gravito-dyons (dyons) and vice versa.

2 Quaternionic Unified Charge of Dyons and Gravito-Dyons

Let us describe the property of quaternion algebra with the use of natural units ($c = \hbar = G = 1$) in order to reformulate the unified theory of generalized electromagnetic fields (associated with dyons) and generalized gravito-Heavisidian fields (associated with gravito-dyons) of linear gravity. So we define the unified charge [60, 61] as

$$Q = (e, g, m, h) = e - ig - jm - kh, \tag{1}$$

where e, g, m and h are respectively described as the electric, magnetic, gravitational and Heavisidian charges (masses). In (1), i, j and k are the quaternion units satisfy the following properties

$$ij = -ji = k(say) \tag{2}$$

and

$$\begin{aligned} i(ij) &= (ii)j = -j, \\ (ij)j &= i(jj) = -i, \\ ik &= -ki = -j, \\ kj &= -jk = -i, \\ i^2 &= j^2 = k^2 = -1. \end{aligned} \tag{3}$$

Complex structure (e, g) represents the generalized charge of dyons of electromagnetic fields while (m, h) denotes the generalized charge of gravito-dyons. The norm of unified quaternion charge (1) is

$$N(Q) = Q\overline{Q} = (e^2 + g^2 + m^2 + h^2), \tag{4}$$

where

$$\overline{Q} = (e, -g, -m, -h) = e + ig + jm + kh. \tag{5}$$

Unified quaternion valued four-potential may then be defined as

$$\{V_\mu\} = \{A_\mu\} - i\{B_\mu\} - j\{C_\mu\} - k\{D_\mu\}, \tag{6}$$

where $\{A_\mu\}$ is the four-potential associated with the dynamics of electric charge, $\{B_\mu\}$ is used for magnetic charge, $\{C_\mu\}$ describes the gravitational charge (mass) while $\{D_\mu\}$ has been associated with the gravi-magnetic (Heavisidian) charge (mass). Then the various potentials of (6) are written in the following quaternionic forms,

$$\begin{aligned} A &= A_0 - iA_1 - jA_2 - kA_3, \\ B &= B_0 - iB_1 - jB_2 - kB_3, \\ C &= C_0 - iC_1 - jC_2 - kC_3, \\ D &= D_0 - iD_1 - jD_2 - kD_3. \end{aligned} \tag{7}$$

Similarly one may define [61] the quaternion valued unified field tensor as

$$\Im_{\mu\nu} = F_{\mu\nu} - iM_{\mu\nu} - jf_{\mu\nu} - kN_{\mu\nu}, \tag{8}$$

where

$$\begin{aligned} F_{\mu\nu} &= A_{\mu,\nu} - A_{\nu,\mu} - i\varepsilon_{\mu\nu\rho\sigma} B^{\rho\sigma}, \\ M_{\mu\nu} &= B_{\mu,\nu} - B_{\nu,\mu} - i\varepsilon_{\mu\nu\rho\sigma} A^{\rho\sigma}, \\ f_{\mu\nu} &= C_{\mu,\nu} - C_{\nu,\mu} - i\varepsilon_{\mu\nu\rho\sigma} D^{\rho\sigma}, \\ N_{\mu\nu} &= D_{\mu,\nu} - D_{\nu,\mu} - i\varepsilon_{\mu\nu\rho\sigma} C^{\rho\sigma} \end{aligned} \tag{9}$$

are the field tensors respectively associated with the dynamics of the electric, magnetic, gravitational and Heavisidian charges (masses). These field tensors satisfy the following Maxwellian field equations

$$\begin{aligned} F_{\mu\nu,\nu} &= j_\mu^{(e)}, \\ M_{\mu\nu,\nu} &= j_\mu^{(m)}, \\ f_{\mu\nu,\nu} &= j_\mu^{(G)}, \\ N_{\mu\nu,\nu} &= j_\mu^{(H)}, \end{aligned} \tag{10}$$

where $j_\mu^{(e)}$ is the four-current associated with electric charge, $j_\mu^{(m)}$ is the four-current associated with magnetic charge, $j_\mu^{(G)}$ is the four-current for gravitational charge(mass) while $j_\mu^{(H)}$ is the four-current for Heavisidian charge (mass). In (8)–(10), the field tensors $F_{\mu\nu}$ and $M_{\mu\nu}$ are associated with dyons, while $f_{\mu\nu}$ and $N_{\mu\nu}$ are associated with gravito-dyons. As such the quaternion valued current may then be defined as

$$J_\mu = j_\mu^{(e)} - i j_\mu^{(m)} - j j_\mu^{(G)} - k j_\mu^{(H)}. \tag{11}$$

3 Lagrangian Formulation and Field Equations

The Lagrangian density for the unified charged particle containing the electric, magnetic, gravitational and Heavisidian charges and the rest mass of the unified particle M, may be written in the following form

$$\begin{aligned} L = {} & -M - \frac{1}{4}\left[\alpha\left\{(A_{\mu,\nu} - A_{\nu,\mu})^2 - (B_{\mu,\nu} - B_{\nu,\mu})^2 - (C_{\mu,\nu} - C_{\nu,\mu})^2 - (D_{\mu,\nu} - D_{\nu,\mu})^2\right\}\right] \\ & - 2\beta\left[\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\left(B_{\mu,\nu} - B_{\nu,\mu}\right) + \left(C_{\mu,\nu} - C_{\nu,\mu}\right)\left(D_{\mu,\nu} - D_{\nu,\mu}\right)\right] \\ & - 2\gamma\left[\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\left(C_{\mu,\nu} - C_{\nu,\mu}\right) + \left(B_{\mu,\nu} - B_{\nu,\mu}\right)\left(D_{\mu,\nu} - D_{\nu,\mu}\right)\right] \\ & - 2\Delta\left[\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\left(D_{\mu,\nu} - D_{\nu,\mu}\right) + \left(B_{\mu,\nu} - B_{\nu,\mu}\right)\left(C_{\mu,\nu} - C_{\nu,\mu}\right)\right] \\ & + \left(\alpha A_\mu - \beta B_\mu + \gamma C_\mu - \Delta D_\mu\right) j_\mu^{(e)} - \left(\alpha B_\mu - \beta A_\mu + \gamma D_\mu - \Delta C_\mu\right) j_\mu^{(m)} \\ & + \left(\alpha C_\mu - \beta D_\mu + \gamma A_\mu - \Delta B_\mu\right) j_\mu^{(G)} - \left(\alpha D_\mu - \beta C_\mu + \gamma B_\mu - \Delta A_\mu\right) j_\mu^{(H)} \\ = {} & L_P + L_F + L_I, \end{aligned} \tag{12}$$

where L_P is the free particle Lagrangian, L_F is the field Lagrangian and L_I is the interaction Lagrangian and $\alpha, \beta, \gamma, \ \Delta$ are real positive arbitrary unimodular parameters satisfying the following conditions

$$\alpha - i\beta - j\gamma - k\Delta = e^{-\theta\hat{n}} = \cos\theta - \hat{n}\sin\theta, \tag{13}$$

and

$$\alpha + i\beta + j\gamma + k\Delta = e^{\theta\hat{n}} = \cos\theta + \hat{n}\sin\theta. \tag{14}$$

From (13) and (14), we get

$$\alpha^2 + \beta^2 + \gamma^2 + \Delta^2 = 1. \tag{15}$$

As such, we may write the constancy condition [47–50] as,

$$-\tan\theta = \frac{g}{e} = \frac{h}{m} = \frac{B_\mu}{A_\mu} = \frac{D_\mu}{C_\mu} = \frac{j_\mu^{(m)}}{j_\mu^{(e)}} = \frac{j_\mu^{(H)}}{j_\mu^{(G)}}. \tag{16}$$

The action integral of the above unified system may be written as,

$$S = \int L dt = S_P + S_F + S_I, \tag{17}$$

where the action part S_P depends upon the properties of the particle, S_F depends on the properties of the field and S_I depends on the parameters of the particle and field both.

In deriving the equation of motion of the particle, we vary the trajectory of the particle without changing the field parameters and as such the action S_F does not affect the motion. On the other hand, in order to find the field equations we take the variation with respect to field parameters assuming the trajectory of the particle fixed. For deriving the equation of motion, we write the concerned part of action in the following form,

$$\begin{aligned} S &= S_P + S_I \\ &= -\int_{t_1}^{t_2} M dt + \int_{t_1}^{t_2} \left(\alpha A_\mu - \beta B_\mu + \gamma C_\mu - \Delta D_\mu\right) j_0^{(e)} \frac{dx_\mu}{dt} dt \\ &\quad - \int_{t_1}^{t_2} \left(\alpha B_\mu - \beta A_\mu + \gamma D_\mu - \Delta C_\mu\right) j_0^{(m)} \frac{dx_\mu}{dt} dt \\ &\quad + \int_{t_1}^{t_2} \left(\alpha C_\mu - \beta D_\mu + \gamma A_\mu - \Delta B_\mu\right) j_0^{(G)} \frac{dx_\mu}{dt} dt \\ &\quad - \int_{t_1}^{t_2} \left(\alpha D_\mu - \beta C_\mu + \gamma B_\mu - \Delta A_\mu\right) j_0^{(H)} \frac{dx_\mu}{dt} dt, \end{aligned} \tag{18}$$

where $j_\mu^{(e)}$, $j_\mu^{(m)}$, $j_\mu^{(G)}$ and $j_\mu^{(H)}$ are expressed in terms of $j_0^{(e)}$, $j_0^{(m)}$, $j_0^{(G)}$ and $j_0^{(H)}$ and $\frac{dx_\mu}{dt}$. Equation (18) may also be written as,

$$\begin{aligned} S = -M\int_a^b dS &+ \int_a^b \left(\alpha A_\mu - \beta B_\mu + \gamma C_\mu - \Delta D_\mu\right) j_0^{(e)} dx_\mu \\ &- \int_a^b \left(\alpha B_\mu - \beta A_\mu + \gamma D_\mu - \Delta C_\mu\right) j_0^{(m)} dx_\mu \\ &+ \int_a^b \left(\alpha C_\mu - \beta D_\mu + \gamma A_\mu - \Delta B_\mu\right) j_0^{(G)} dx_\mu \end{aligned}$$

$$-\int_a^b \left(\alpha D_\mu - \beta C_\mu + \gamma B_\mu - \Delta A_\mu\right) j_0^{(H)} dx_\mu, \tag{19}$$

where the first term is an integral along the world line of the particle between two events a and b, i.e. the presence of the particle at its initial and final positions at time t_1 and t_2. The variation of the action S may be written as,

$$\begin{aligned}\delta S = \delta\Bigg\{ \int_a^b -M dS + \int_a^b \Big[&\left(\alpha A_\mu - \beta B_\mu + \gamma C_\mu - \Delta D_\mu\right) j_0^{(e)} dx_\mu \\ &- \left(\alpha B_\mu - \beta A_\mu + \gamma D_\mu - \Delta C_\mu\right) j_0^{(m)} dx_\mu + \left(\alpha C_\mu - \beta D_\mu + \gamma A_\mu - \Delta B_\mu\right) j_0^{(G)} dx_\mu \\ &- \left(\alpha D_\mu - \beta C_\mu + \gamma B_\mu - \Delta A_\mu\right) j_0^{(H)} dx_\mu \Big] \Bigg\} = 0.\end{aligned} \tag{20}$$

Taking the variation of the terms one by one, we get

$$I = \delta \int_a^b -M dS = \int_a^b M u_\mu d\delta x_\mu, \tag{21}$$

where $u_\mu = \frac{dx_\mu}{dS}$ (four-velocity). Integrationg equation (21) by parts, we get

$$I = \left| M u_\mu \delta x_\mu \right|_a^b - \int_a^b M du_\mu \delta x_\mu, \tag{22}$$

where first term is zero, since the integral is varied between fixed limits,

$$(\delta x_\mu)_b = (\delta x_\mu)_a = 0. \tag{23}$$

Thus

$$I = -\int_a^b M du_\mu \delta x_\mu = -\int_a^b M \frac{du_\mu}{dS} dS \delta x_\mu. \tag{24}$$

Now it is convenient to take the following variations in order to solve the equations of motion of varius charges,

$$\begin{aligned}\delta \int_a^b \alpha A_\mu j_0^{(e)} dx_\mu &= \int_a^b \alpha j_0^{(e)} (A_{\nu,\mu} - A_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \alpha j_0^{(e)} F_{\mu\nu} u_\nu \delta x_\mu dS; \\ \delta \int_a^b \beta B_\mu j_0^{(e)} dx_\mu &= \int_a^b \beta j_0^{(e)} (B_{\nu,\mu} - B_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \beta j_0^{(e)} M_{\mu\nu} u_\nu \delta x_\mu dS; \\ \delta \int_a^b \gamma C_\mu j_0^{(e)} dx_\mu &= \int_a^b \gamma j_0^{(e)} (C_{\nu,\mu} - C_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \gamma j_0^{(e)} f_{\mu\nu} u_\nu \delta x_\mu dS; \\ \delta \int_a^b \Delta D_\mu j_0^{(e)} dx_\mu &= \int_a^b \Delta j_0^{(e)} (D_{\nu,\mu} - D_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \Delta j_0^{(e)} N_{\mu\nu} u_\nu \delta x_\mu dS;\end{aligned} \tag{25}$$

$$\begin{aligned}
\delta\int_a^b \alpha B_\mu j_0^{(m)} dx_\mu &= \int_a^b \alpha j_0^{(m)} (B_{\nu,\mu} - B_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \alpha j_0^{(m)} M_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \beta A_\mu j_0^{(m)} dx_\mu &= \int_a^b \beta j_0^{(m)} (A_{\nu,\mu} - A_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \beta j_0^{(m)} F_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \gamma D_\mu j_0^{(m)} dx_\mu &= \int_a^b \gamma j_0^{(m)} (D_{\nu,\mu} - D_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \gamma j_0^{(m)} N_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \Delta C_\mu j_0^{(m)} dx_\mu &= \int_a^b \Delta j_0^{(m)} (C_{\nu,\mu} - C_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \Delta j_0^{(m)} f_{\mu\nu} u_\nu \delta x_\mu dS;
\end{aligned} \tag{26}$$

$$\begin{aligned}
\delta\int_a^b \alpha C_\mu j_0^{(G)} dx_\mu &= \int_a^b \alpha j_0^{(G)} (C_{\nu,\mu} - C_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \alpha j_0^{(G)} f_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \beta D_\mu j_0^{(G)} dx_\mu &= \int_a^b \beta j_0^{(G)} (D_{\nu,\mu} - D_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \beta j_0^{(G)} N_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \gamma A_\mu j_0^{(G)} dx_\mu &= \int_a^b \gamma j_0^{(G)} (A_{\nu,\mu} - A_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \gamma j_0^{(G)} F_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \Delta B_\mu j_0^{(G)} dx_\mu &= \int_a^b \Delta j_0^{(G)} (B_{\nu,\mu} - B_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \Delta j_0^{(G)} M_{\mu\nu} u_\nu \delta x_\mu dS;
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
\delta\int_a^b \alpha D_\mu j_0^{(H)} dx_\mu &= \int_a^b \alpha j_0^{(H)} (D_{\nu,\mu} - D_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \alpha j_0^{(H)} N_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \beta C_\mu j_0^{(H)} dx_\mu &= \int_a^b \beta j_0^{(H)} (C_{\nu,\mu} - C_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \beta j_0^{(H)} f_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \gamma A_\mu j_0^{(H)} dx_\mu &= \int_a^b \gamma j_0^{(H)} (A_{\nu,\mu} - A_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \gamma j_0^{(H)} F_{\mu\nu} u_\nu \delta x_\mu dS;\\
\delta\int_a^b \Delta B_\mu j_0^{(H)} dx_\mu &= \int_a^b \Delta j_0^{(H)} (B_{\nu,\mu} - B_{\mu,\nu}) u_\nu \delta x_\mu dS = \int_a^b \Delta j_0^{(H)} M_{\mu\nu} u_\nu \delta x_\mu dS.
\end{aligned} \tag{28}$$

Substituting (24)–(28) in (20), we get the following equations of motion for the dynamics of fields of a particle containing four charges as,

$$\begin{aligned}
M\frac{d^2 x_\mu}{dt^2} &= eF_{\mu\nu} u^\nu;\\
M\frac{d^2 x_\mu}{dt^2} &= gM_{\mu\nu} u^\nu;\\
M\frac{d^2 x_\mu}{dt^2} &= mf_{\mu\nu} u^\nu;\\
M\frac{d^2 x_\mu}{dt^2} &= hN_{\mu\nu} u^\nu.
\end{aligned} \tag{29}$$

Here M is the effective mass given by

$$M = m - \frac{\kappa - 1}{2} h, \tag{30}$$

where $\kappa = +1$ for electromagnetic fields and $\kappa = -1$ for gravito-Heavisidian fields. The field equations (29) may also be obtained from the action (17) by taking the trajectory of the unified particle fixed and considering the variation of the field parameters (i.e. potentials) only. So, the parameters associated unified charged particle such as the free particle action and four-current density J_μ are treated to be constant i.e. $\delta S_P = 0$. From (12), we may write the field and interaction parts of the Lagrangian required for the variation as,

$$\begin{aligned} L_F + L_I &= -\frac{1}{4}\left[\alpha(F_{\mu\nu}F^{\mu\nu} - M_{\mu\nu}M^{\mu\nu} - f_{\mu\nu}f^{\mu\nu} - N_{\mu\nu}N^{\mu\nu})\right] \\ &\quad - 2\beta\left[F_{\mu\nu}M_{\mu\nu} + f_{\mu\nu}N_{\mu\nu}\right] - 2\gamma\left[F_{\mu\nu}f_{\mu\nu} + M_{\mu\nu}N_{\mu\nu}\right] - 2\Delta\left[F_{\mu\nu}N_{\mu\nu} + f_{\mu\nu}M_{\mu\nu}\right] \\ &\quad + \left(\alpha A_\mu - \beta B_\mu + \gamma C_\mu - \Delta D_\mu\right) j_\mu^{(e)} - \left(\alpha B_\mu - \beta A_\mu + \gamma D_\mu - \Delta C_\mu\right) j_\mu^{(m)} \\ &\quad + \left(\alpha C_\mu - \beta D_\mu + \gamma A_\mu - \Delta B_\mu\right) j_\mu^{(G)} - \left(\alpha D_\mu - \beta C_\mu + \gamma B_\mu - \Delta A_\mu\right) j_\mu^{(H)}. \end{aligned} \tag{31}$$

Now considering the term wise variation in (31), we get

$$\begin{aligned} \delta S_1 &= \int \frac{-\alpha}{4}\delta F_{\mu\nu}^2 d\Omega = \int \frac{-\alpha}{2} F_{\mu\nu}\delta F_{\mu\nu} d\Omega; \\ \delta S_2 &= \int \frac{-\alpha}{4}\delta M_{\mu\nu}^2 d\Omega = \int \frac{-\alpha}{2} M_{\mu\nu}\delta M_{\mu\nu} d\Omega; \\ \delta S_3 &= \int \frac{-\alpha}{4}\delta f_{\mu\nu}^2 d\Omega = \int \frac{-\alpha}{2} f_{\mu\nu}\delta f_{\mu\nu} d\Omega; \\ \delta S_4 &= \int \frac{-\alpha}{4}\delta N_{\mu\nu}^2 d\Omega = \int \frac{-\alpha}{2} N_{\mu\nu}\delta N_{\mu\nu} d\Omega; \end{aligned} \tag{32}$$

$$\begin{aligned} \delta S_5 &= \int \frac{\beta}{2}\delta(F_{\mu\nu}M_{\mu\nu} + f_{\mu\nu}N_{\mu\nu}) d\Omega \\ &= \int \frac{\beta}{2}\left\{\left[F_{\mu\nu}\delta M_{\mu\nu} + M_{\mu\nu}\delta F_{\mu\nu}\right] + \left[f_{\mu\nu}\delta N_{\mu\nu} + N_{\mu\nu}\delta f_{\mu\nu}\right]\right\} d\Omega; \end{aligned} \tag{33}$$

$$\begin{aligned} \delta S_6 &= \int \frac{\gamma}{2}\delta(F_{\mu\nu}f_{\mu\nu} + M_{\mu\nu}N_{\mu\nu}) d\Omega \\ &= \int \frac{\gamma}{2}\left\{\left[F_{\mu\nu}\delta f_{\mu\nu} + f_{\mu\nu}\delta F_{\mu\nu}\right] + \left[M_{\mu\nu}\delta N_{\mu\nu} + N_{\mu\nu}\delta M_{\mu\nu}\right]\right\} d\Omega; \end{aligned} \tag{34}$$

$$\begin{aligned} \delta S_7 &= \int \frac{\Delta}{2}\delta(F_{\mu\nu}N_{\mu\nu} + M_{\mu\nu}f_{\mu\nu}) d\Omega \\ &= \int \frac{\Delta}{2}\left\{\left[F_{\mu\nu}\delta N_{\mu\nu} + N_{\mu\nu}\delta F_{\mu\nu}\right] + \left[M_{\mu\nu}\delta f_{\mu\nu} + f_{\mu\nu}\delta M_{\mu\nu}\right]\right\} d\Omega; \end{aligned} \tag{35}$$

$$\begin{aligned}\delta S_8 =& \int \alpha j_\mu^{(e)}\delta A_\mu - \int \beta j_\mu^{(e)}\delta B_\mu + \int \gamma j_\mu^{(e)}\delta C_\mu - \int \Delta j_\mu^{(e)}\delta D_\mu \\ & - \int \alpha j_\mu^{(m)}\delta B_\mu + \int \beta j_\mu^{(m)}\delta A_\mu - \int \gamma j_\mu^{(m)}\delta D_\mu + \int \triangle j_\mu^{(m)}\delta C_\mu \\ & + \int \alpha j_\mu^{(G)}\delta C_\mu - \int \beta j_\mu^{(G)}\delta D_\mu + \int \gamma j_\mu^{(G)}\delta A_\mu - \int \Delta j_\mu^{(G)}\delta D_\mu \\ & - \int \alpha j_\mu^{(H)}\delta D_\mu + \int \beta j_\mu^{(H)}\delta C_\mu - \int \gamma j_\mu^{(H)}\delta B_\mu + \int \triangle j_\mu^{(H)}\delta A_\mu. \end{aligned} \tag{36}$$

δS_1 may be calculated after integrating it by parts and applying Gauss theorem as

$$\delta S_1 = -\alpha \int \frac{\partial F_{\mu\nu}}{\partial x_\mu}\delta A_\mu d\Omega. \tag{37}$$

Similarly, other variations may be calculated. taking into account the equation

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4 + \delta S_5 + \delta S_6 + \delta S_7 + \delta S_8 = 0 \tag{38}$$

from which we get

$$\begin{aligned} & \int \left(j_\mu^{(e)}\delta A_\mu + j_\mu^{(m)}\delta B_\mu + j_\mu^{(G)}\delta C_\mu + j_\mu^{(H)}\delta D_\mu\right)\alpha d\Omega \\ & \quad - \int \left(\frac{\partial F_{\mu\nu}}{\partial x_\mu}\delta A_\mu - \frac{\partial M_{\mu\nu}}{\partial x_\mu}\delta B_\mu - \frac{\partial f_{\mu\nu}}{\partial x_\mu}\delta C_\mu - \frac{\partial N_{\mu\nu}}{\partial x_\mu}\delta D_\mu\right)\alpha d\Omega \\ & \quad - \int \left(j_\mu^{(e)}\delta B_\mu + j_\mu^{(m)}\delta A_\mu + j_\mu^{(G)}\delta D_\mu + j_\mu^{(H)}\delta C_\mu\right)\alpha d\Omega \\ & \quad - \int \left(\frac{\partial F_{\mu\nu}}{\partial x_\mu}\delta A_\mu - \frac{\partial M_{\mu\nu}}{\partial x_\mu}\delta B_\mu - \frac{\partial f_{\mu\nu}}{\partial x_\mu}\delta C_\mu - \frac{\partial N_{\mu\nu}}{\partial x_\mu}\delta D_\mu\right)\alpha d\Omega \\ & \quad - \int (j_\mu^{(e)} - i j_\mu^{(m)} - j j_\mu^{(G)} - k j_\mu^{(H)})\delta(A_\mu + iB_\mu + jC_\mu + kD_\mu)\beta d\Omega \\ & \quad - \int (F_{\mu\nu,\nu} - iM_{\mu\nu,\nu} - jf_{\mu\nu,\nu} - kN_{\mu\nu,\nu})\delta(A_\mu + iB_\mu + jC_\mu + kD_\mu)\beta d\Omega \\ & \quad - \int (j_\mu^{(e)} - i j_\mu^{(m)} - j j_\mu^{(G)} - k j_\mu^{(H)})\delta(B_\mu + iA_\mu + jD_\mu + kC_\mu)\beta d\Omega \\ & \quad - (F_{\mu\nu,\nu} - iM_{\mu\nu,\nu} - jf_{\mu\nu,\nu} - kN_{\mu\nu,\nu})\delta(B_\mu + iA_\mu + jD_\mu + kC_\mu)\beta d\Omega \\ & = 0 \end{aligned} \tag{39}$$

which yields the Maxwellian field equations given by (10) for the dynamics of electric, magnetic, gravitational and Heavisidian charges (masses). These equations thus provide the following form of unified field equations of a particle simultaneously contains these four charges (masses)

$$\Im_{\mu\nu,\nu} = J_\mu, \tag{40}$$

where $\Im_{\mu\nu}$ is the unified field tensor defined by (8) and $J_\mu$is the unified current density given by (11).

4 Equation of Motion of a Unified Charged in Quaternionic Form

From (26), we may write the unified equation of motion for a particle simultaneously contains four charges (masses) in terms of quaternion as

$$M\frac{d^2x_\mu}{dt^2} = \left\{Re\left(\bar{Q}\Im_{\mu\nu}\right)\right\}u^\nu \tag{41}$$

where $\ddot{x_\mu} = \frac{d^2x_\mu}{dt^2}$ is particle acceleration, u^ν is the four-velocity, and $\bar{Q}$ is the quaternion conjugate of the unified charge given by (5). The right hand side of (41) may also be written as

$$\left\{Re\left(\bar{Q}\Im_{\mu\nu}\right)\right\}u^\nu = \left(eF_{\mu\nu} + gM_{\mu\nu} + mf_{\mu\nu} + hN_{\mu\nu}\right)u^\nu \tag{42}$$

which can be reduced as

$$\begin{aligned} Re\left[Q, \Im_{\mu\nu}\right]u^\nu &= \frac{1}{2}\left(Q\bar{\Im}_{\mu\nu} + \bar{Q}\Im_{\mu\nu}\right)u^\nu \\ &= \left(eF_{\mu\nu} + gM_{\mu\nu} + mf_{\mu\nu} + hN_{\mu\nu}\right)u^\nu \\ &= \left\{e\left[\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{H}\right] + g\left[\overrightarrow{H} - \overrightarrow{v} \times \overrightarrow{E}\right]\right\} \\ &\quad + \left\{m\left[\overrightarrow{G} - \overrightarrow{v} \times \overrightarrow{\mathcal{H}}\right] + h\left[\overrightarrow{\mathcal{H}} + \overrightarrow{v} \times \overrightarrow{G}\right]\right\}, \end{aligned} \tag{43}$$

where $\overrightarrow{E}$ is the electric field, $\overrightarrow{H}$ is the magnetic field, $\overrightarrow{G}$ is the gravitational field and $\overrightarrow{\mathcal{H}}$ is Heavisidian field.

5 Euler's Lagrangian Equation of Motion of Unified Charged

The Lagrangian density for unified fields of a particle containing simultaneously the four charges namely electric, magnetic, gravitational and Heavisidian may also be expressed as

$$\begin{aligned} L = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}N_{\mu\nu}N^{\mu\nu} \\ &+ A_\mu j_\mu^{(e)} + B_\mu j_\mu^{(m)} + C_\mu j_\mu^{(G)} + D_\mu j_\mu^{(H)}. \end{aligned} \tag{44}$$

This Lagrangian density may also be written in terms of Grassmann product as

$$L = \frac{1}{8}\left[\Im_{\mu\nu}, \bar{\Im}_{\mu\nu}\right] + \left[V_\mu, \bar{J}_\mu\right], \tag{45}$$

where $[V_\mu, \bar{J}_\mu] = \frac{1}{2}[V_\mu\bar{J}_\mu + \bar{V}_\mu J_\mu]$, with $\bar{V}_\mu$ (the quaternion conjugate of the unified four-potential), $\bar{J}_\mu$ (the quaternion conjugate of unified four-current density) and $\bar{\Im}_{\mu\nu}$ (the quaternion conjugate of unified field tensor) are defined as follows,

$$\begin{aligned} \bar{V}_\mu &= A_\mu + iB_\mu + jC_\mu + kD_\mu; \\ \bar{J}_\mu &= j_\mu^{(e)} + ij_\mu^{(m)} + jj_\mu^{(G)} + kj_\mu^{(H)}; \\ \bar{\Im}_{\mu\nu} &= F_{\mu\nu} + iM_{\mu\nu} + jf_{\mu\nu} + kN_{\mu\nu}. \end{aligned} \tag{46}$$

Then

$$\left[V_\mu, \bar{J}_\mu\right] = \left[A_\mu j_\mu^{(e)} + B_\mu j_\mu^{(m)} + C_\mu j_\mu^{(G)} + D_\mu j_\mu^{(H)}\right]. \tag{47}$$

Similarly, we have

$$\frac{1}{8}\left[\Im_{\mu\nu}, \bar{\Im}_{\mu\nu}\right] = \frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu} + M_{\mu\nu}M^{\mu\nu} + f_{\mu\nu}f^{\mu\nu} + N_{\mu\nu}N^{\mu\nu}\right]. \tag{48}$$

From the Lagrangian density (44), we get the Euler's Lagrangian equations in the following forms;

$$\begin{aligned}
\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu A_\mu)} &= 0; \\
\frac{\partial L}{\partial B_\mu} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu B_\mu)} &= 0; \\
\frac{\partial L}{\partial C_\mu} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu C_\mu)} &= 0; \\
\frac{\partial L}{\partial D_\mu} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu D_\mu)} &= 0.
\end{aligned} \tag{49}$$

This equation provides the field equations associated respectively with the dynamics of electric, magnetic, gravitational and Heavisidian charges (masses) given by (10) after taking care the usual method of variations with respect to potential. These equation may immediately be generalized to (40) as unified field equations of a particle simultaneously containing four charges namely electric, magnetic, gravitational and Heavisidian charges (masses).

6 Conclusion

A unified Lagrangian density of generalized electromagnetic fields and Heavisidian fields have been developed with the fact that these fields essentially possess the structural symmetry. The action integral (17) of the unified quaternionic charge (i.e. the complex structure of dyons and gravito-dyons) in the field of others constructed by choosing the suitable Lagrangian density (12), does not make the use of string variables. Rather, it has been written in terms of four-potentials associated with four charges namely electric, magnetic, gravitational and Heavisidian charges. Following the usual method of variation, it is shown that this action leads to equation of motion (29) and unified field equations (40). The beauty of the Lagrangian (12) lies in the fact that individual components of four-currents and four-potentials have been embodied in the corresponding generalized quantities through the unknown parameters α, β, γ and Δ which satisfy the constraints described by (13, 14) and (15). It has already been shown that the Lagrangian density (12), equation of motion (29) and the unified field equations (40) are invariant under the rotation in charge space or its combination with space and time reflections and also under the reflection in charge space combined with time reversal or space reflection. It means that the system possess strong symmetry under rotation in charge space.

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